A Note on a Conjecture on Permanents

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ABSTRACT

Let K_n denote the set of all $n \times n$ nonnegative matrices whose entries have sum n, and let φ be a real function on K_n defined by $\varphi(X) = \prod_{i=1}^{n} \sum_{j=1}^{n} x_{ij} + \prod_{j=1}^{n} \sum_{i=1}^{n} x_{ij} - \operatorname{per} X$ for $X = [x_{ij}] \in K_n$. A matrix $A \in K_n$ is called a φ-maximizing matrix on K_n if $\varphi(A) \geqslant \varphi(X)$ for all $X \in K_n$. It is conjectured that $J_n = [1/n]_{n \times n}$ is the unique φ-maximizing matrix on K_n. In this note, the following are proved: (i) If A is a positive φ-maximizing matrix, then $A = J_n$. (ii) If A is a row stochastic φ-maximizing matrix, then $A = J_n$. (iii) Every row sum and every column sum of a φ-maximizing matrix lies between $1 - \sqrt{2 \cdot n!/n^n}$ and $1 + (n-1)\sqrt{2 \cdot n!/n^n}$. (iv) For any p.s.d. symmetric $A \in K_n$, $\varphi(A) \leqslant 2 - n!/n^n$ with equality iff $A = J_n$. (v) φ attains a strict local maximum on K_n at J_n.

1. INTRODUCTION

An $n \times n$ real nonnegative matrix is called a *row* (*column*) *stochastic* matrix if all the row sums (column sums) are equal to 1. A matrix which is both row stochastic and column stochastic is called a *doubly stochastic* matrix. Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices, and let J_n be the $n \times n$ matrix all of whose entries are $1/n$. As usual, the permanent of a matrix A will be denoted by $\operatorname{per} A$. For the minimization of the permanent on the set Ω_n, we have a

THEOREM (van der Waerden and Egoryčev [1]). *For any $A \in \Omega_n$,*

$$\operatorname{per} A \geqslant \frac{n!}{n^n},$$

with equality if and only if $A = J_n$.

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Throughout this note, let K_n denote the set of all $n \times n$ real nonnegative matrices $X = [x_{ij}]$ with

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} = n,$$

and let φ denote a real valued function on K_n defined by

$$\varphi(X) = \prod_{i=1}^{n} \sum_{j=1}^{n} x_{ij} + \prod_{j=1}^{n} \sum_{i=1}^{n} x_{ij} - \operatorname{per} X$$

for $X = [x_{ij}] \in K_n$. A matrix $A \in K_n$ will be called a *φ-maximizing matrix* on K_n if $\varphi(A) \geqslant \varphi(X)$ for all $X \in K_n$. For each $k = 1, 2, \ldots,$ let $\delta_k = k!/k^k$.

Among several problems and conjectures concerning the permanent function, there is one which is apparently due to E. Dittert:

CONJECTURE [2, Conjecture 28]. For any $X \in K_n$,

$$\varphi(X) \leqslant 2 - \delta_n,$$

with equality if and only if $X = J_n$.

Recently, R. Sinkhorn [4] has proved that a φ-maximizing matrix on K_n has a positive permanent, and that the conjecture is true for $n = 2$.

In this note we investigate some properties of φ-maximizing matrices on K_n which may support the conjecture and settle the conjecture for a subclass of K_n, namely:

(i) If A is a positive φ-maximizing matrix on K_n, then $A = J_n$.

(ii) If A is a row stochastic φ-maximizing matrix on K_n, then $A = J_n$.

(iii) If A is a φ-maximizing matrix with ith row sum r_i and jth column sum c_j, $i, j = 1, \ldots, n$, then

$$1 - \sqrt{2\delta_n} < r_i, c_j < 1 + (n-1)\sqrt{2\delta_n}$$

for all $i, j = 1, \ldots, n$.

(iv) For any p.s.d. symmetric $A \in K_n$, $\varphi(A) \leqslant 2 - \delta_n$, with equality if and only if $A = J_n$.

(v) φ attains a strict local maximum on K_n at J_n.

For a matrix A, $A(i|j)$ denotes the matrix obtained from A by deleting the ith row and the jth column.

2. φ-MAXIMIZING MATRICES

The following lemma forms a basis for an averaging process under certain circumstances.

LEMMA 1. *Let $A = [a_{ij}]$ be a φ-maximizing matrix on K_n with row sum vector $(r_1, \ldots, r_n)$, column sum vector $(c_1, \ldots, c_n)$, and let $1 \leqslant s, t \leqslant n$ be such that $s \neq t$. Then*

(i) $\operatorname{per} A(i|s) - \operatorname{per} A(i|t) = \left(\dfrac{1}{c_s} - \dfrac{1}{c_t}\right) \prod_{j=1}^{n} c_j$ *if $a_{is} > 0$ and $a_{it} > 0$,*

(ii) $\operatorname{per} A(i|s) - \operatorname{per} A(i|t) \geqslant \left(\dfrac{1}{c_s} - \dfrac{1}{c_t}\right) \prod_{j=1}^{n} c_j$ *if $a_{is} = 0$ and $a_{it} > 0$.*

Proof. (i): Suppose that $a_{is} > 0$ and $a_{it} > 0$. Let ε be a real number such that $|\varepsilon| < \min\{a_{is}, a_{it}\}$, and let $A_\varepsilon = A + \varepsilon(E_{is} - E_{it})$, where E_{ij} stands for an n-square matrix all of whose entries are 0 except for the (i, j) entry, which is 1. Then, clearly, $A_\varepsilon \in K_n$ and A_ε has row sum vector $(r_1, \ldots, r_n)$ and column sum vector $(c_1', \ldots, c_n')$ where $c_s' = c_s + \varepsilon$, $c_t' = c_t - \varepsilon$, and $c_i' = c_i$ for $i \neq s, t$. We also have

$$\operatorname{per} A_\varepsilon = \operatorname{per} A + [\operatorname{per} A(i|s) - \operatorname{per} A(i|t)]\, \varepsilon.$$

Let

$$r = \prod_{i=1}^{n} r_i \quad \text{and} \quad c = \prod_{i=1}^{n} c_i.$$

Then

$$\begin{aligned}
\varphi(A_\varepsilon) &= r + \frac{c}{c_s c_t}(c_s + \varepsilon)(c_t - \varepsilon) - \operatorname{per} A_\varepsilon \\
&= r + \frac{c}{c_s c_t}\left[c_s c_t + (c_t - c_s)\varepsilon\right] - \operatorname{per} A \\
&\quad - [\operatorname{per} A(i|s) - \operatorname{per} A(i|t)]\,\varepsilon + O(\varepsilon^2) \\
&= \varphi(A) + \left[c\left(\frac{1}{c_s} - \frac{1}{c_t}\right) - [\operatorname{per} A(i|s) - \operatorname{per} A(i|t)]\right]\varepsilon + O(\varepsilon^2).
\end{aligned} \tag{1}$$

Thus we get

$$c\left(\frac{1}{c_s}-\frac{1}{c_t}\right)-[\operatorname{per}A(i|s)-\operatorname{per}A(i|t)]=0,$$

completing the proof of (i).

(ii): Suppose that $a_{is}=0$ and $a_{it}>0$. If we choose ε such that $0<\varepsilon<a_{it}$, then $A_\varepsilon \in K_n$. Now, from (1), it follows that

$$c\left(\frac{1}{c_s}-\frac{1}{c_t}\right)-[\operatorname{per}A(i|s)-\operatorname{per}A(i|t)]\leqslant 0$$

by the maximality of A, which completes the proof of (ii). ■

Indeed, the above lemma can be slightly generalized to

(i)′ $\operatorname{per}A(i|s)-\operatorname{per}A(k|t)=r\left(\frac{1}{r_i}-\frac{1}{r_k}\right)+c\left(\frac{1}{c_s}-\frac{1}{c_t}\right)$ if $a_{is}>0,\ a_{kt}>0$,

(ii)′ $\operatorname{per}A(i|s)-\operatorname{per}A(k|t)\geqslant r\left(\frac{1}{r_i}-\frac{1}{r_k}\right)-c\left(\frac{1}{c_s}-\frac{1}{c_t}\right)$ if $a_{is}=0,\ a_{kt}>0$,

where r and c are as in the proof of Lemma 1(i). We can give a similar proof by using $A_\varepsilon=A+\varepsilon(E_{is}-E_{kt})$.

Lemma 2. *Let $A=[a_{ij}]$, $(r_1,\dots,r_n)$, $(c_1,\dots,c_n)$, and s,t be the same as in Lemma 1. If*

$$\operatorname{per}A(i|s)-\operatorname{per}A(i|t)=\left(\frac{1}{c_s}-\frac{1}{c_t}\right)\prod_{j=1}^{n}c_j$$

whenever $a_{is}+a_{it}>0$, then the matrix obtained from A by replacing each of the columns s and t with their average is also a φ-maximizing matrix on K_n. A similar statement holds for rows.

Proof. Let B be the new matrix gotten from A by averaging columns s and t, and let

$$r=\prod_{i=1}^{n}r_i,\qquad c=\prod_{i=1}^{n}c_i,\quad\text{and}\quad c\left(\frac{1}{c_s}-\frac{1}{c_t}\right)=h,$$

so that $\operatorname{per} A(i|s) = \operatorname{per} A(i|t) + h$. Then

$$\begin{aligned}\operatorname{per} B &= \frac{1}{4}\left[2\operatorname{per} A + \sum_{i=1}^{n} a_{is}\operatorname{per} A(i|t) + \sum_{i=1}^{n} a_{it}\operatorname{per} A(i|s)\right]\\ &= \frac{1}{4}\left[2\operatorname{per} A + \sum_{i=1}^{n} a_{is}[\operatorname{per} A(i|s) - h] + \sum_{i=1}^{n} a_{it}[\operatorname{per} A(i|t) + h]\right]\\ &= \frac{1}{4}\left(4\operatorname{per} A - \sum_{i=1}^{n} a_{is}h + \sum_{i=1}^{n} a_{it}h\right)\\ &= \operatorname{per} A + \tfrac{1}{4}(c_s - c_t)h,\end{aligned}$$

and

$$\varphi(B) = r + \frac{c}{c_s c_t}\left(\frac{c_s + c_t}{2}\right)^2 - \operatorname{per} A - \tfrac{1}{4}(c_s - c_t)h.$$

Therefore

$$\begin{aligned}\varphi(B) - \varphi(A) &= \frac{1}{4}\frac{c}{c_s c_t}(c_s + c_t)^2 - c - \tfrac{1}{4}(c_s - c_t)c\left(\frac{1}{c_s} - \frac{1}{c_t}\right)\\ &= \frac{c}{4c_s c_t}\left[(c_s - c_t)^2 - (c_s - c_t)^2\right] = 0,\end{aligned}$$

i.e. $\varphi(A) = \varphi(B)$, implying that B is a φ-maximizing matrix on K_n. ∎

Corollary 2.1. *Let $A = [a_1, \dots, a_n]$ be a φ-maximizing matrix on K_n, and let $2 \leqslant p \leqslant n$. If $a_1, \dots, a_p$ have the same* (0,1)*-pattern, then $A(J_p \oplus I_{n-p})$ is a φ-maximizing matrix on K_n. A similar statement holds for rows.*

Proof. Let $M_p = (J_2 \oplus I_{p-2})(I_1 \oplus J_2 \oplus I_{p-3}) \cdots (I_{p-2} \oplus J_2)$. Then, by Lemmas 1 and 2, $A(M_p \oplus I_{n-p})$ is a φ-maximizing matrix on K_n. Now $A(J_p \oplus I_{n-p}) = \lim_{k\to\infty} A(M_p \oplus I_{n-p})^k$ is a φ-maximizing matrix on K_n, since the set of all φ-maximizing matrices is a compact set. ∎

Now, we are ready to prove the following

THEOREM 1. *If A is a positive φ-maximizing matrix on K_n, then*

$$\varphi(A) = 2 - \frac{n!}{n^n} \quad \text{and} \quad A = J_n.$$

Proof. By Corollary 2.1, $J_n = J_n A J_n$ is a φ-maximizing matrix on K_n. Thus $\varphi(A) = \varphi(J_n) = 2 - n!/n^n$. To show $A = J_n$, first consider the following n-square matrix

$$X_t = \begin{bmatrix} 1-(n-1)t & t & \cdots & t \\ \vdots & \vdots & & \vdots \\ 1-(n-1)t & t & \cdots & t \end{bmatrix},$$

where $0 \leqslant t \leqslant 1/(n-1)$. Clearly, $X_t \in K_n$. Define a real function f on the interval $[0, 1/(n-1)]$ by $f(t) = \varphi(X_t)$. Then

$$\begin{aligned} f(t) &= n^n[1-(n-1)t]t^{n-1} + 1 - n![1-(n-1)t]t^{n-1} \\ &= (n^n - n!)\left[t^{n-1} - (n-1)t^n\right] + 1. \end{aligned}$$

Thus

$$\begin{aligned} f'(t) &= (n^n - n!)\left[(n-1)t^{n-2} - n(n-1)t^{n-1}\right] \\ &= (n^n - n!)(n-1)t^{n-2}(1-nt), \end{aligned}$$

which tells us that

$$f(t) < f\left(\frac{1}{n}\right), \qquad \text{i.e.} \quad \varphi(X_t) < \varphi(J_n),$$

for all t such that $0 \leqslant t \leqslant 1/(n-1)$ and $t \neq 1/n$. Now, we are about to show that A is doubly stochastic.

Let $(c_1, \ldots, c_n)$ be the column sum vector of A. Without loss of generality, we may assume that $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n$. Notice that $c_1 \geqslant 1$, and also that if $c_1 = 1$, then $c_1 = \cdots = c_n = 1$. Suppose that $c_1 > 1$. Let $B = J_n A(I_1 \oplus J_{n-1})$;

then B is a φ-maximizing matrix on K_n by Corollary 2.1. On the other hand, $B = X_t$, with

$$0 < t = \frac{n - c_1}{n(n-1)} < \frac{n-1}{n(n-1)} = \frac{1}{n}.$$

Thus, it follows, by the above observation for the function f, that $\varphi(B) < \varphi(J_n)$, contradicting the maximality of B. Therefore it must be that $c_1 = 1$ and hence $c_1 = \cdots = c_n = 1$. Similarly that all the row sums of A are 1 can be proved. So A is a doubly stochastic matrix. Thus we conclude that $A = J_n$ by the van der Waerden–Egoryčev theorem. ■

To prove our next lemma, we shall use Alexandrov's inequality for permanents:

$$(\operatorname{per} A)^2 \geqslant \operatorname{per}[\mathbf{a}_1, \mathbf{a}_1, \mathbf{a}_3, \dots, \mathbf{a}_n] \operatorname{per}[\mathbf{a}_2, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n]$$

for any nonnegative n-square matrix $A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$, which is a reformulation of the original one due to Egoryčev [1].

Lemma 3. *Let A be a φ-maximizing matrix on K_n with the column sum vector $(c_1, \dots, c_n)$, and let $1 \leqslant s < t \leqslant n$. If $c_s = c_t$, then the matrix obtained from A by replacing each of the columns s and t with their average is also a φ-maximizing matrix on K_n. A similar statement holds for rows.*

Proof. Without loss of generality, we may assume (by permuting rows, if necessary) that $A = [a_{ij}]$ is of the form

$$A = \begin{array}{c} \begin{array}{ccccc} & s & & t & \\ & \downarrow & & \downarrow & \end{array} \\ \begin{bmatrix} & a_{1s} & & 0 & \\ & \vdots & & \vdots & \\ & a_{ps} & & 0 & \\ & 0 & & a_{p+1,t} & \\ & \vdots & & \vdots & \\ & 0 & & a_{qt} & \\ * & & * & & * \\ & a_{q+1,s} & & a_{q+1,t} & \\ & \vdots & & \vdots & \\ & a_{ks} & & a_{kt} & \\ & 0 & & 0 & \\ & \vdots & & \vdots & \\ & 0 & & 0 & \end{bmatrix} \end{array},$$

where $0 \leqslant p \leqslant q \leqslant k \leqslant n$, and

$$a_{is} > 0 \quad \text{if and only if} \quad i = 1, \dots, p, q+1, \dots, k,$$

$$a_{it} > 0 \quad \text{if and only if} \quad i = p+1, \dots, k.$$

If $q = 0$, then columns s and t have the same (0,1)-pattern, and hence the lemma is verified by Corollary 2.1. Suppose that $q \geqslant 1$. Then, by Lemma 1, there is a nonnegative q-vector $(h_1, \dots, h_q)$ such that

$$\operatorname{per} A(i|s) - \operatorname{per} A(i|t) = \begin{cases} -h_i, & 1 \leqslant i \leqslant p, \\ h_i, & p+1 \leqslant i \leqslant q. \end{cases}$$

Notice also that $\operatorname{per} A(i|s) = \operatorname{per} A(i|t)$ for $i = q+1, \dots, k$. Now, Alexandrov's inequality implies

$$\begin{aligned}
(\operatorname{per} A)^2 &\geqslant \left[\sum_{i=1}^{n} a_{is} \operatorname{per} A(i|t)\right]\left[\sum_{i=1}^{n} a_{it} \operatorname{per} A(i|s)\right] \\
&= \left[\sum_{i=1}^{p} a_{is}(\operatorname{per} A(i|s) + h_i) + \sum_{i=p+1}^{n} a_{is} \operatorname{per} A(i|s)\right] \\
&\quad \times \left[\left(\sum_{i=1}^{p} + \sum_{i=q+1}^{n}\right) a_{it} \operatorname{per} A(i|t) + \sum_{i=p+1}^{q} a_{it}(\operatorname{per} A(i|t) + h_i)\right] \\
&= \left(\operatorname{per} A + \sum_{i=1}^{p} a_{is} h_i\right)\left(\operatorname{per} A + \sum_{i=p+1}^{q} a_{it} h_i\right) \\
&\geqslant (\operatorname{per} A)^2 + \left(\sum_{i=1}^{p} a_{is} h_i + \sum_{i=p+1}^{q} a_{it} h_i\right) \operatorname{per} A.
\end{aligned}$$

Since $a_{is} > 0$ $(i = 1, \dots, p)$, $a_{it} > 0$ $(i = p+1, \dots, q)$ and (by Sinkhorn [3]) $\operatorname{per} A > 0$, it must be that $h_1 = \cdots = h_q = 0$. Thus we have proved that $\operatorname{per} A(i|s) = \operatorname{per} A(i|t)$ for $i = 1, \dots, k$. Now the conclusion of the lemma follows from Lemma 2. ■

COROLLARY 3.1. *Let A be a φ-maximizing matrix on K_n with column sum vector $(c_1, \dots, c_n)$, and let $2 \leqslant p \leqslant n$. If $c_1 = \cdots = c_p$, then $A(J_p \oplus I_{n-p})$ is a φ-maximizing matrix on K_n. A similar statement holds for rows.*

THEOREM 2. *If A is a row stochastic φ-maximizing matrix on K_n, then $\varphi(A)=2-\delta_n$ and $A=J_n$.*

Proof. Let $B=J_nA$. Then, by Corollary 3.2, B is a φ-maximizing matrix on K_n. Since no column sum of A is 0, B is a positive matrix. Hence $B=J_n$ by Theorem 1. Therefore $\varphi(A)=\varphi(B)=2-n!/n^n$. Moreover, since A and $B=J_n$ have the same column sum vector, we see that A is, in fact, a doubly stochastic matrix. Thus it follows that $A=J_n$. ■

Because of Theorem 2, to prove the conjecture it suffices to show that every φ-maximizing matrix is row (or column) stochastic. In the next theorem, we shall show that every φ-maximizing matrix is one which is close to a doubly stochastic matrix, in the sense that the row and column sums approach 1 as $n\to\infty$.

THEOREM 3. *Let A be a φ-maximizing matrix on K_n with row sum vector $(r_1,\dots,r_n)$ and column sum vector $(c_1,\dots,c_n)$. Then*

$$1-\sqrt{2\delta_n}<r_i,c_j<1+(n-1)\sqrt{2\delta_n}$$

for all i, $j=1,\dots,n$.

Proof. Without loss of generality, we may assume that $r_1\leqslant r_2\leqslant\cdots\leqslant r_n$. Then $r_1\leqslant 1$, with equality if and only if $r_2=\cdots=r_n=1$. So, in the case that $r=1$, the theorem for row sums is readily verified. Suppose $r_1<1$. Then

$$\prod_{i=2}^{n} r_i \leqslant \left(\frac{n-r_1}{n-1}\right)^{n-1}=\left(1+\frac{1-r_1}{n-1}\right)^{n-1}<e^{1-r_1}. \tag{1}$$

Since A is a φ-maximizing matrix on K_n, we have

$$\prod_{i=1}^{n} r_i+\prod_{j=1}^{n} c_j-\operatorname{per}A\geqslant 2-\delta_n. \tag{2}$$

By (1), we also have

$$\prod_{i=1}^{n} r_i+\prod_{j=1}^{n} c_j-\operatorname{per}A\leqslant r_1\prod_{i=2}^{n} r_i+1<r_1e^{1-r_1}+1. \tag{3}$$

Now from (2) and (3), it follows that

$$r_1 e^{1-r_1} > 1 - \delta_n,$$

which is the same as

$$(1-h)\sum_{k=0}^{\infty}\frac{h^k}{k!} > 1-\delta_n,$$

where $h = 1 - r_1 > 0$, that is,

$$1-\sum_{k=2}^{\infty}\left[\frac{1}{(k-1)!}-\frac{1}{k!}\right]h^k > 1-\delta_n.$$

Hence

$$\frac{1}{2}h^2+\sum_{k=3}^{\infty}\left[\frac{1}{(k-1)!}-\frac{1}{k!}\right]h^k < \delta_n,$$

implying that $h < \sqrt{2\delta_n}$, i.e. that $r_1 > 1 - \sqrt{2\delta_n}$ as required.

Now

$$r_n = n - \sum_{i=1}^{n-1} r_i < n-(n-1)\left(1-\sqrt{2\delta_n}\right) = 1+(n-1)\sqrt{2\delta_n}.$$

Thus we have proved that

$$1-\sqrt{2\delta_n} < r_i < 1+(n-1)\sqrt{2\delta_n} \qquad (i=1,\ldots,n).$$

Similarly for the c_j's. ■

In the following, we shall prove the conjecture for certain classes of matrices in K_n.

LEMMA 4 [3]. *Let $A = [a_{ij}]$ be a positive semidefinite Hermitian complex matrix of order n. Then*

$$\operatorname{per} A \geqslant \frac{n!}{s(A)^n}\prod_{i=1}^{n}|r_i|^2$$

where $r_i = \sum_{j=1}^{n} a_{ij}$ $(i=1,\ldots,n)$ and $s(A) = \sum_{i=1}^{n} r_i \neq 0$.

THEOREM 4. *Let $A \in K_n$ be positive semidefinite symmetric. Then $\varphi(A) \leqslant 2-\delta_n$, with equality if and only if $A = J_n$.*

Proof. By Lemma 4, we have

$$\varphi(A) = 2r - \operatorname{per} A \leqslant 2r - \delta_n r^2,$$

where $r = r_1 \cdots r_n$, and r_i is the ith row sum of A for $i = 1,\dots,n$. Since

$$\frac{d}{dr}\left(2r - \delta_n r^2\right) = 2(1-\delta_n r) > 0$$

for all r, $0 \leqslant r \leqslant 1$, $n \geqslant 2$, we see that

$$2r - \delta_n r^2 \leqslant 2 - \delta_n,$$

with equality if and only if $r = 1$. But $r = 1$ implies that $r_1 = \cdots = r_n = 1$, i.e. that A is doubly stochastic and $\operatorname{per} A = \delta_n$. Thus, it follows that $A = J_n$ by the van der Waerden–Egoryčev theorem. ■

Let $\operatorname{Bd}(K_n)$ denote the boundary of K_n, i.e. the set of all matrices of K_n having at least one zero entry.

THEOREM 5. *The function φ attains a strict local maximum on K_n at J_n.*

Proof. Let $X = [x_{ij}] \in \operatorname{Bd}(K_n)$, and let $R = (r_1,\dots,r_n)$ and $C = (c_1,\dots,c_n)$ be the row and column sum vectors of X respectively. Let e denote the n-vector $(1,\dots,1)$. For a real number t, let $X_t = tX + (1-t)J_n$. Then the row and column sum vectors of X_t are $(1-t)\mathrm{e} + tR$ and $(1-t)\mathrm{e} + tC$ respectively. Define the functions g, h, p, and f by

$$g(t) = \prod_{i=1}^{n} \left(1 + (r_i - 1)t\right),$$

$$h(t) = \prod_{i=1}^{n} \left(1 + (c_i - 1)t\right),$$

$$p(t) = \operatorname{per} X_t,$$

and

$$f(t) = g(t) + h(t) - p(t) = \varphi(X_t)$$

on the closed unit interval $[0,1]$. Then

$$g'(t)=g(t)\sum_{i=1}^{n}\frac{r_i-1}{1+(r_i-1)t}$$

and

$$g''(t)=g(t)\left[\left(\sum_{i=1}^{n}\frac{r_i-1}{1+(r_i-1)t}\right)^2-\sum_{i=1}^{n}\left(\frac{r_i-1}{1+(r_i-1)t}\right)^2\right],$$

which imply that

$$g'(0)=0 \quad \text{and} \quad g''(0)=-\|R-\mathbf{e}\|^2,$$

where $\|\cdot\|$ stands for the Euclidean norm. Similarly we can show that

$$h'(0)=0 \quad \text{and} \quad h''(0)=-\|C-\mathbf{e}\|^2.$$

For the function p, we have

$$p'(t)=\sum_{i,j}\left(x_{ij}-\frac{1}{n}\right)\operatorname{per}X_t(i|j)$$

and

$$p''(t)=\sum_{i,j}\left(x_{ij}-\frac{1}{n}\right)\sum_{u\neq i}\sum_{v\neq j}\left(x_{uv}-\frac{1}{n}\right)\operatorname{per}X_t(i,u|j,v).$$

By a sequence of elementary computations we can show that $p'(0)=0$ and

$$\begin{aligned}p''(0)&=\frac{(n-2)!}{n^{n-2}}\left[\sum_{i,j}\left(x_{ij}-\frac{1}{n}\right)^2-\sum_{i,j}\left(x_{ij}-\frac{1}{n}\right)(r_i+c_j-2)\right]\\&=\frac{(n-2)!}{n^{n-2}}\left(\|X-J_n\|^2-\|R-\mathbf{e}\|^2-\|C-\mathbf{e}\|^2\right)\end{aligned}$$

by computing first that

$$\sum_{i,j}\left(x_{ij}-\frac{1}{n}\right)(r_i+c_j)=\|R-\mathbf{e}\|^2+\|C-\mathbf{e}\|^2.$$

Therefore we have shown that $f'(0)=0$ and

$$\begin{aligned} f''(0) &= \left(\frac{(n-2)!}{n^{n-2}}-1\right)\left(\|R-\mathbf{e}\|^2+\|C-\mathbf{e}\|^2\right) \\ &\quad -\frac{(n-2)!}{n^{n-2}}\|X-J_n\|^2 \\ &< -\frac{(n-2)!}{n^{n-2}}\frac{1}{n^2} = -\frac{(n-2)!}{n^n} < 0 \end{aligned}$$

since $x_{ij}=0$ for at least one (i,j). Now the conclusion of the theorem follows from the fact that K_n is a convex set whose boundary $\mathrm{Bd}(K_n)$ is compact. ■

REMARK. If A is a φ-maximizing matrix on K_n, then

$$0<\operatorname{per}A\leqslant\delta_n.$$

That $0<\operatorname{per}A$ is due to Sinkhorn [4]. For the upper bound of $\operatorname{per}A$, if, on the contrary, $\operatorname{per}A>\delta_n$, then

$$2-\delta_n\leqslant\prod_{i=1}^n r_i+\prod_{i=1}^n c_i-\operatorname{per}A$$

implies that

$$2<2+\operatorname{per}A-\delta_n\leqslant\prod_{i=1}^n r_i+\prod_{i=1}^n c_i\leqslant 2,$$

where r_i and c_j, $i,j=1,\ldots,n$, are respectively the ith row sum and jth column sum of A, which is impossible. If, in particular, $\operatorname{per}A=\delta_n$, then, necessarily,

$$\prod_{i=1}^n r_i+\prod_{j=1}^n c_j=2$$

implying that $r_1 = \cdots = r_n = c_1 = \cdots = c_n = 1$, i.e. that A is doubly stochastic.

In view of the above remark and Theorem 2, we see that the followings are equivalent.

(i) Dittert's conjecture: J_n is the unique φ-maximizing matrix on K_n.
(ii) Every φ-maximizing matrix on K_n is row (or column) stochastic.
(iii) Every φ-maximizing matrix on K_n has permanent $n!/n^n$.

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